

Decoherence Free Subspace and entanglement by interaction with a common squeezed bath.

D. Mundarain¹ and M. Orszag²

¹ *Departments de Física, Universidad Simon Bolvar, Apartado Postal 89000, Caracas 1080A, Venezuela,* ² *Facultad de Física, Pontificia Universidad Católica de Chile, Casilla 306, Santiago, Chile*

In this work we find explicitly the decoherence free subspace (DFS) for a two two-level system in a common squeezed vacuum bath. We also find an orthogonal basis for the DFS composed of a symmetrical and an antisymmetrical (under particle permutation) entangled state. For any initial symmetrical state, the master equation has one stationary state which is the symmetrical entangled decoherence free state. In this way, one can generate entanglement via common squeezed bath of the two systems. If the initial state does not have a definite parity, the stationary state depends strongly on the initial conditions of the system and it is a statistical mixture of states which belong to DFS. We also study the effect of the coupling between the two-level systems on the DFS.

One of the most important challenges of Quantum Computation is to revert the effects of the environment over systems used to store information. In the general case, these interactions limit the capability of a quantum computer [1], [2], [3], [4]. There are some interesting proposal to circumvent the injurious influence of reservoirs, one of them is related to the use of Decoherence Free Subspace (DFS) as the memory space for storing the quantum information. The search of ways to bypass decoherence in Quantum Information Processing (QIP) started with Palma *et al* [5], in a study of the dephasing of two qubits in contact with a reservoir. Duan and Gao [6], used this model with a different method and coined the term "Coherence preserving states". The general framework for DFS was done by Zanardi *et al* [7] in a spin-boson model, in a "collective decoherence mode", undergoing both dephasing and dissipation. Based on dynamic symmetries in the spin-environment interaction, they provide a general condition for the decoherence free states.

Also, Lidar *et al* [8] introduced the term "decoherence-free subspaces" (DFS) and analyzed the robustness of these states against perturbations. A completely general condition for the existence of the DF states in terms of the Kraus operators was provided by Lidar, Bacon and Whaley [9]. A simple definition of the DFS, is the following one: *a system with Hilbert space H is said to have a decoherence free subspace \tilde{H} if the evolution inside $\tilde{H} \subset H$ is purely unitary.*

In the presence of the environment, the DFS is a set of all states which are not affected at all by the interaction with the bath. In terms of the reduced dynamics of the system, they are invariant states. Since not all quantum open systems have a DFS, it is an important issue to study the systems which do and also investigate the dynamical properties of such systems.[10], [11], [12], [13]

In this letter we are concerned with a two two-level system interacting with a common squeezed vacuum bath. We show that this system has a DFS. We will review some properties of the stationary states of this system and will establish the relations between them and the DFS. We

will also address another important issue: the preparation of entangled states. We will show that for any initial symmetrical state, state which is invariant under particle permutation, the stationary state depends on the squeezing parameters of the bath, and most importantly, it is pure and entangled.

The problem of the stationary states of a master equation, or expressed more mathematically, the stationary states of quantum dynamical semigroups is, in general, an involved problem.[14] In our case, we derive directly from the master equation a set of linear differential equations which are solved explicitly in order to obtain the stationary states of the system. The stationary states depend strongly on the initial conditions and they are statistical mixtures of states which belong to the DFS.

Lets first consider the master equation, in the interaction picture, for a two level system in a broadband squeezed vacuum [15]:

$$\begin{aligned} \frac{\partial \rho}{\partial t} = & \frac{1}{2}\gamma(N+1)(2\sigma\rho\sigma^\dagger - \sigma^\dagger\sigma\rho - \rho\sigma^\dagger\sigma) \\ & + \frac{1}{2}\gamma N(2\sigma^\dagger\rho\sigma - \sigma\sigma^\dagger\rho - \rho\sigma\sigma^\dagger) \\ & - \frac{1}{2}\gamma M e^{i\psi}(2\sigma^\dagger\rho\sigma^\dagger - \sigma^\dagger\sigma^\dagger\rho - \rho\sigma^\dagger\sigma^\dagger) \\ & - \frac{1}{2}\gamma M e^{-i\psi}(2\sigma\rho\sigma - \sigma\sigma\rho - \rho\sigma\sigma) \end{aligned} \quad (1)$$

where γ is the vacuum decay constant and $N, M = \sqrt{N(N+1)}$ and ψ are the squeeze parameters of the bath. The two Pauli spin matrices are:

$$\sigma^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (2)$$

It is not difficult to show that the master equation can be written in an explicit Lindblad form using only one Lindblad operator:

$$\frac{\partial \rho}{\partial t} = \frac{\gamma}{2} \{2S\rho S^\dagger - \rho S^\dagger S - S^\dagger S\rho\} \quad (3)$$

where

$$S = \sqrt{N+1}\sigma - \sqrt{N}\exp\{i\psi\}\sigma^\dagger \quad (4)$$

$$= \cosh(r)\sigma - \sinh(r)\exp\{i\psi\}\sigma^\dagger \quad (5)$$

This operator has two eigenstates:

$$S|\lambda_\pm\rangle = \lambda_\pm|\lambda_\pm\rangle \quad (6)$$

where

$$|\lambda_\pm\rangle = \sqrt{\frac{N}{N+M}}|+\rangle \mp i\sqrt{\frac{M}{N+M}}e^{-i\psi/2}|-\rangle \quad (7)$$

with eigenvalues $\lambda_\pm = \pm i\sqrt{M}\exp\{i\psi/2\}$.

For a two two-level system interacting with a common squeezed bath, the master equation has the same structure as in the one particle case, but now:

$$S = \sqrt{N+1}\Sigma - \sqrt{N}\exp\{i\psi\}\Sigma^\dagger \quad (8)$$

where the Σ 's are the combined ladder operators for the two particles:

$$\Sigma = \sigma_1 + \sigma_2, \quad \Sigma^\dagger = \sigma_1^\dagger + \sigma_2^\dagger \quad (9)$$

Thus, the two particle Lindblad operator can be written as the sum of the two Lindblad operators of each particle:

$$S = S_1 + S_2 \quad (10)$$

In this case the DFS [9] is composed of all eigenstates of S with zero eigenvalues. The following two linearly independent states satisfy this property.

$$|\psi_1\rangle = |\lambda_+\rangle_1 \otimes |\lambda_-\rangle_2 \quad (11)$$

$$|\psi_2\rangle = |\lambda_-\rangle_1 \otimes |\lambda_+\rangle_2 \quad (12)$$

where the subscripts 1,2 refer to the particle one and two respectively. These two states are not orthogonal. They define a plane in the Hilbert space, spanned by the following orthonormal states:

$$|\phi_1\rangle = \frac{1}{\sqrt{N^2+M^2}}(N|+,+\rangle + Me^{-i\psi}|-,-\rangle), \quad (13)$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|-,+\rangle - |+,-\rangle), \quad (14)$$

where $|\pm, \pm\rangle = |\pm\rangle_1 \otimes |\pm\rangle_2$ and $|\pm, \mp\rangle = |\pm\rangle_1 \otimes |\mp\rangle_2$.

Now, one can define two other states which are orthogonal to $|\phi_1\rangle$ and $|\phi_2\rangle$:

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|-,+\rangle + |+,-\rangle), \quad (15)$$

$$|\phi_4\rangle = \frac{1}{\sqrt{N^2+M^2}}(M|+,+\rangle - Ne^{-i\psi}|-,-\rangle). \quad (16)$$

In the basis $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle, |\phi_4\rangle\}$ the Lindblad operator S for the two particles has a very simple form:

$$s = \begin{pmatrix} 0 & 0 & \alpha e^{i\psi} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta e^{i\psi} \\ 0 & 0 & \beta & 0 \end{pmatrix} \quad (17)$$

where

$$\alpha = \sqrt{\frac{2}{2N+1}}, \quad (18)$$

$$\beta = -2\sqrt{N(N+1)}\alpha, \quad (19)$$

$$\delta = \frac{2}{\alpha}. \quad (20)$$

In this basis the master equation becomes a system of fifteen differential equations. The various components of the system are either constant or exponentially decaying terms with rates that depend on α , β and δ . The exponentially decaying terms go to zero and eventually one finds the following stationary state for the master equation:

$$\rho(\infty) = \rho_{ss} = \begin{pmatrix} 1 - \rho_{22}(0) & \rho_{12}(0) & 0 & 0 \\ \rho_{21}(0) & \rho_{22}(0) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (21)$$

The first important thing to notice from this equation is that when the initial state does not contain the anti-symmetrical state $|\phi_2\rangle$ ($\rho_{22}(0) = \rho_{12}(0) = \rho_{21}(0) = 0$) the stationary state of the master equation is the pure entangled state $|\phi_1\rangle$:

$$\rho_{ss} = |\phi_1\rangle\langle\phi_1| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (22)$$

In figure (1) we show the time evolution of the probability $\langle\phi_1|\rho(t)|\phi_1\rangle$ for the initial symmetrical state $|++\rangle$. As one can see this quantity goes to 1 as $t \rightarrow \infty$. This fact indicates that the final state is the decoherence free state $|\phi_1\rangle$. This result also shows that one can generate entanglement via a common squeezed bath of the two two-level systems.

In any other case, the stationary states depends on the initial states of the system, but in general it is a mixed of state *that belongs to the DFS of the master equation*.

$$\rho_{ss} = P_1|\nu_1\rangle\langle\nu_1| + P_2|\nu_2\rangle\langle\nu_2| \quad (23)$$

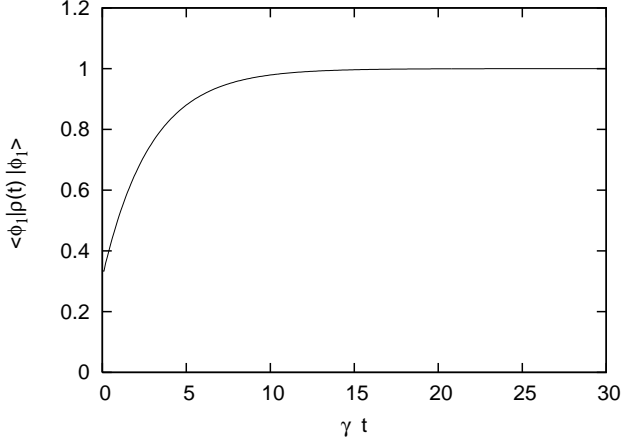


FIG. 1: Probability for the system to be in $|\phi_1\rangle$ as a function of time. We took $N = 1$, $\psi = 0$, and the initial state is $|++\rangle$. One starts with a factorized state and generates an entangled one via common squeezed bath.

where

$$|\nu_1\rangle = A_1 ((P_1 - \rho_{22}(0))|\phi_1\rangle + \rho_{21}(0)|\phi_2\rangle) \quad (24)$$

$$|\nu_2\rangle = A_2 ((P_2 - \rho_{22}(0))|\phi_1\rangle + \rho_{21}(0)|\phi_2\rangle) \quad (25)$$

where the normalization constants A_1, A_2 are

$$A_{1,2} = \frac{1}{\sqrt{|\rho_{12}(0)|^2 - (P_{1,2} - \rho_{22}(0))^2}} \quad (26)$$

and

$$P_{1,2} = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + |\rho_{12}(0)|^2 - \rho_{22}(0)(1 - \rho_{22}(0))} \quad (27)$$

One can quantify the purity of the final states observing that as

$$\text{Tr}(\rho_{ss}^2) \leq 1 \quad (28)$$

then

$$|\rho_{12}(0)|^2 \leq \rho_{22}(0)(1 - \rho_{22}(0)) \quad (29)$$

From this expression we define a parameter x which characterizes the initial coherence between the states $|\phi_1\rangle$ and $|\phi_2\rangle$:

$$|\rho_{12}(0)|^2 = x\rho_{22}(0)(1 - \rho_{22}(0)). \quad (30)$$

From the initial considerations this parameter also characterizes the purity of final state. From this we obtain three cases in which the final states are pure:

1. $\rho_{22}(0) = 0$, which corresponds to the case of having a symmetrical initial state; the stationary state is $|\phi_1\rangle$.

2. $\rho_{22}(0) = 1$ which corresponds to the case of having $|\phi_2\rangle$ as initial state; this state does not evolve because it belongs to the DFS. It is an invariant state.
3. $\rho_{22}(0) \neq 0$ or 1 and $x = 1$, in this case the initial state must be a pure state which is a linear combination of $|\phi_1\rangle$ and $|\phi_2\rangle$. This state is also an invariant state.

In all other cases, the stationary state is a mixed state, but it remains entangled.

We consider the interesting limit $N \rightarrow 0$ (vacuum bath). In this limit, the states $|\phi_1\rangle$ and $|\phi_4\rangle$ of the orthonormal basis become:

$$|\phi_1\rangle = |--\rangle \quad \text{and} \quad |\phi_4\rangle = |++\rangle \quad (31)$$

and the other two states of the basis do not change. The stationary state has the same previous structure with the new basis. Any symmetrical state decays to $|\phi_1\rangle = |--\rangle$. In particular

$$|++\rangle \rightarrow |--\rangle \quad (32)$$

It is interesting to observe that when the initial state is not completely symmetrical, the system does not decay to the $|--\rangle$ state. For example, for the initial condition:

$$|--\rangle = \frac{1}{\sqrt{2}}(|\phi_2\rangle + |\phi_3\rangle) \quad (33)$$

one has:

$$\rho_{22}(0) = \frac{1}{2} \quad \text{and} \quad \rho_{12}(0) = 0 \quad (34)$$

and we get the following result:

$$|--\rangle \rightarrow \frac{1}{2}|\phi_1\rangle\langle\phi_1| + \frac{1}{2}|\phi_2\rangle\langle\phi_2| \quad (35)$$

For a higher dimensional problem, for instance, when the number of spins $\mathcal{N}=4$ we can form the products

$$|\psi_1\rangle = |\lambda_+\rangle_1 \otimes |\lambda_+\rangle_2 \otimes |\lambda_-\rangle_3 \otimes |\lambda_-\rangle_4, \quad (36)$$

and cyclic permutations (6 possible combinations), which belong to the DFS since we have the same number of equal positive and negative eigenvalues of S , adding up to zero.

For even \mathcal{N} , the dimension of the DFS is

$$\text{DIM}(\text{DFS of } \mathcal{N} \text{ spins}) = \frac{\mathcal{N}!}{\left[\left(\frac{\mathcal{N}}{2}\right)!\right]^2}. \quad (37)$$

In general, in order to have 2 atoms in a common bath, they will have to be quite near, at a distance no bigger than the correlation length of the bath. This implies that one cannot avoid an interaction between the particles. This interaction between the two-level systems can, in

principle, affect the DFS. For example, one can consider a Dipole-Dipole Van der Waals coupling of the form:

$$H_D = \hbar\Omega(\sigma_1\sigma_2^\dagger + \sigma_1^\dagger\sigma_2), \quad (38)$$

with $\Omega = |\mathbf{d}|^2 \frac{(1-3\cos^2\theta)}{R^3}$, where R is the modulus of the distance between the atoms and θ the angle between the separation vector and \mathbf{d} (dipole matrix).

It is interesting to study the effect of such a Hamiltonian over our decoherence free states $|\phi_1\rangle$ and $|\phi_2\rangle$. It is simple to verify that:

$$\begin{aligned} (\sigma_1\sigma_2^\dagger + \sigma_1^\dagger\sigma_2)|\phi_1\rangle &= 0, \\ (\sigma_1\sigma_2^\dagger + \sigma_1^\dagger\sigma_2)|\phi_2\rangle &= -|\phi_2\rangle. \end{aligned} \quad (39)$$

As we can see, for an initial state within the DFS, with this type of coupling the state remains within the DFS. As a matter of fact, the time evolution operator just introduces a time dependent phase factor in $|\phi_2\rangle$ and leaves $|\phi_1\rangle$ invariant:

$$\begin{aligned} |\phi_2\rangle &\rightarrow \exp[-i\Omega(\sigma_1\sigma_2^\dagger + \sigma_1^\dagger\sigma_2)t]|\phi_2\rangle = \exp[i\Omega t]|\phi_2\rangle, \\ |\phi_1\rangle &\rightarrow \exp[-i\Omega(\sigma_1\sigma_2^\dagger + \sigma_1^\dagger\sigma_2)t]|\phi_1\rangle = |\phi_1\rangle. \end{aligned} \quad (40)$$

Thus, the dipole-dipole coupling does not affect the DFS of two two-level systems.

The Ising-type Hamiltonian $H = A \sum_{i=1}^{\mathcal{N}} S_i^z S_{i+1}^z$ for \mathcal{N} two-level systems is another example. It has, again

the following effect: if one starts with a mixed state within the DFS, and since $S^z|\lambda_\pm\rangle = |\lambda_\mp\rangle$, the system will remain in the DFS.

To summarize, we have found a DFS for two two-level system in a common squeezed bath, and found the relation between this subspace and the steady state for any initial condition. In some particular cases, when the initial condition is symmetrical, we get a steady state that is pure and entangled. However, in the most general case, the steady state is in a mixed state within the DFS. This is an interesting property of this particular system, but it is not necessarily true for other systems.

Finally, we calculate the dimension of the general DFS for \mathcal{N} two-level systems, and we also discuss the unavoidable effect of the coupling between these two-level systems.

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